Bachelor of Science
(B.Sc.- PCM)

Real Analysis
(DBSPCO303T24)

Self-Learning Material
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COURSE INTRODUCTION

Real numbers form the backbone of mathematics, serving as the foundation for various mathematical disciplines, including calculus, analysis, and algebra. In this Unit, we will revisit the fundamental concepts of real numbers, exploring their properties, classifications, and significance in mathematical contexts.

The course is divided into 10 units. Each Unit is divided into sub topics. The Units provide students with a comprehensive understanding of the real number system and its characteristics. They also examine continuity, differentiability, and integrability concepts in a rigorous mathematical framework, analyze sequences and series of real numbers and functions, and apply these concepts to solve theoretical and practical problems.

There are sections and sub-sections inside each unit. Each unit starts with a statement of objectives that outlines the goals we hope you will accomplish. Every segment of the unit has many tasks that you need to complete.

We wish you pleasure in the Course. Please try all of the exercises and activities included in the units.

Course Outcomes: After completion of the course, the students will be able to:

1. Recall the many properties of the real line and learn to define sequence in terms of functions from to a subset.
2. Explain bounded, convergent, divergent, Cauchy and monotonic sequences.
3. Apply to calculate their limit superior, limit inferior, and the limit of a bounded sequence.
4. Analyze various applications of the fundamental theorem of integral calculus.
5. Evaluate uniform continuity, differentiation, integration and uniform convergence.
6. Create the ratio, root, and alternating series and limit comparison tests for convergence and absolute convergence of an infinite series of real numbers.

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UNIT-1

Introduction to Real Numbers

Learning Objectives:

- Understanding the Real Number System
- Properties of Real Numbers
- Applications of Real Numbers
- Supremum and Infimum

Structure:

1.1 Review of basic concepts of real numbers:

Real numbers form the backbone of mathematics, serving as the foundation for various mathematical disciplines, including calculus, analysis, and algebra. In this Unit, we will revisit the fundamental concepts of real numbers, exploring their properties, classifications, and significance in mathematical contexts.

Definition 1.1

Real numbers include all rational and irrational numbers and can be represented as points on the real number line. They are denoted by the symbol $\mathbb{R}$.

Properties of Real Numbers:

Real numbers possess several key properties that make them essential in mathematical analysis:

1. **Closure:** The summation, subtraction, and multiplication of two real numbers are also real number.
2. **Commutativity and Associativity:** Addition and multiplication of real numbers are commutative and associative.
3. **Distributive Property:** Multiplication distributes over addition for real numbers.

Ordering: Real numbers can be ordered such that for any two real numbers a and b, either $a < b, a = b, or\ a > b$.

Classification of Real Numbers:

Real numbers can be classified into different categories based on their properties:

i. **Natural Numbers (N):** The set of positive integers, including 1, 2, 3,...
ii. **Whole Numbers (W):** The set of non-negative integers, including 0 and all positive integers.
iii. **Integers (Z):** The set of positive and negative whole numbers, including zero.
iv. **Rational Numbers (Q):**numbers whose fractional representations use two integers and whose denominator does not equal zero.
v. **Irrational Numbers:** Numbers that can't be expressed as a rational of two integers, such as $\sqrt{2}$ and π.

1.2 Countable and uncountable sets:

In the realm of set theory, understanding the distinction between countable and uncountable sets is crucial. These concepts have profound implications in various branches of mathematics, including real analysis, topology, and measure theory. Let's explore these concepts in detail.

Countable Sets:

A set is said to be countable if number of elements are finite.

Ex.$A = \{1, 2, 3\}$

Finite Sets: Finite sets are trivially countable since their elements can be enumerated in a finite sequence.

Countably Infinite Sets: Sets that are infinite but still have a one-to-one correspondence with N are countably infinite. Examples include the set of all integers Z, the set of even integers, and the set of odd integers.

Uncountable Sets:

A set is considered uncountable if its members cannot be put into one-to-one correspondence with the natural numbers. In other words, there is no way to list all the elements of an uncountable set in a sequence.

Real Numbers: The set of real numbers R is a classic example of an uncountable set. This was famously proven by Georg Cantor using his diagonal argument.

Power set: The power set of any set (the set of all its subsets) is always uncountable. This follows from Cantor's theorem.

Cardinality:

Cardinality is a measure of the "size" of a set, indicating the number of elements it contains. Countable sets have cardinality either finite or countably infinite, while uncountable sets have cardinality strictly greater than that of the natural numbers.

Countable Sets: Countable sets have cardinality $\aleph_0$, also known as aleph-null.

Uncountable Sets: Cardinality of uncountable sets is bigger than $\aleph_0$.The cardinality of the real numbers R is denoted by c, and it is strictly greater than $\aleph_0$.

1.3 Real number system:

The real number system is an extensive framework used in mathematics to describe and analyze numbers that can be found on the number line. It includes a variety of subsets with distinct properties and applications. Here's a comprehensive overview of the real number system:

Components of the Real Number System:

1. Natural Numbers (N):

The set of positive integers used for counting. Examples: 1,2,3,…

Properties: Closed under addition and multiplication, but not under subtraction or division.

2. Whole Numbers (W):

The set of natural numbers including zero.Examples: 0,1,2,3,…

Properties: Closed under addition and multiplication.

3. Integers (Z):

The set of whole numbers and their negatives.Examples: …,−3,−2,−1,0,1,2,3,……

Properties: Closed under addition, subtraction, and multiplication, but not under division.

4. Rational Numbers (Q):

Numbers that can be state as a fraction$\frac{a}{b}$, here a and$b$ are integers and b≠0. Examples: 1/2,−4/3,5(since 55 can be written as 5/1)

Properties: Dense in the real number line (between any two rational numbers, there is another rational number).

5. Irrational Numbers:

Numbers that can't be state as a simple fraction. Their decimal expansions are non-terminating and non-repeating. Ex's: π,e,$\sqrt{2}$

Properties: Not closed under addition, subtraction, multiplication, or division (e.g., $\pi + (-\pi) = 0$ which is rational).

Subsets of Real Numbers:

1. Positive and Negative Numbers:

Positive real numbers (R^+): All real numbers greater than zero.

Negative real numbers (R^-): All real numbers less than zero.

2. Non-Negative and Non-Positive Numbers:

Non-negative real numbers: All positive numbers including zero.

<---|---|---|---|---|---|---|---|---|---|---|---|--->

... -3 -2 -1 0 1 2 3 4 5 6 7 ...

Figure 1.1 Real Number Line

Figure 1.1 showing the real line number with $-\infty$ to ∞ which includes the all rational and irrational numbers both.

1.4 Archimedean Property:

Definition 1. 2

The Archimedean property is a fundamental property of the real numbers that can be stated as follows:

"For any two real numbers *x* and *y*with$x > 0$, there exists a natural number *n* such that $nx > y$".

In other words, no matter how large y is or how small *x* is, we can always find a natural number n such that the product nx exceeds y. This property ensures that the real numbers do not have infinitely large or infinitely small values relative to the natural numbers.

Implications and Examples:

1. Unbounded of Natural Numbers:

The Archimedean property implies that the set of natural numbers N is not bounded above in the real numbers R. For any real number *y*, no matter how large, there exists a natural number *n* such that *n*>*y*.

Example: Given $y = 1000$, there exists a natural number n (specifically, $n = 1001$) such that $n > 1000$.

2. Approximation of Real Numbers by Natural Numbers:

For any positive real number *x*, the Archimedean property guarantees that we can find a natural number n such that $1/n < x$. This is useful in analysis for approximations and in constructing sequences that converge to a given limit.

Example: Given $x = 0.001$, there exists a natural number n (specifically, $n = 1000$) such that $1/n < 0.001$.

3. Denseness of Rational Numbers:

The rational numbers are dense in the real numbers as a result of the Archimedean condition. This indicates that there exists a rational number that lies between any two real numbers. The property helps to construct rational approximations to any real number.

Example: For any real numbers a and b with $a < b$, ∃a rational number q:$a < q < b$.

Proof of the Archimedean Property:

Here is a simple proof of the Archimedean property:

Assume for contradiction that the Archimedean property is false.

Then there exist positive real numbers *x* and *y* such that for all natural numbers n, $nx \leq y$.

Consider the sequence $\{xy\}$.

According to our assumption, for all $n \in N$, $n \leq xy$.

This implies that *xy* is an upper bound for the natural numbers.

However, the set of natural numbers N has no upper bound in the real numbers (by definition 1.2).

This contradiction implies that our initial assumption must be false, and thus the Archimedean property holds.

1.5 Supremum, Infimum, and Completeness:

Supremum (Least Upper Bound-LUB):

The sup.of a set *S* of real numbers is the fewest real number that is ≥to every element of *S*.

If *S* is bounded above, the supremum exists and is unique.

Notation: If *S* is a set, then sup*S* denotes the supremum of *S*.

Example: Consider the set $S = \{x \in R \mid 0 \leq x < 1\}$.The supremum of S is 1, since 1 is the smallest number that is ≥every element of *S*.

Infimum (Greatest Lower Bound-GLB):

The inf. of a set *S* of real numbers is the biggest real number that is ≤every element of *S*.

If *S* is bounded below, the infimum exists and is unique.

Notation: If S is a set, then inf$S$ denotes the infimum of S.

Example: Consider the set $S = \{x \in R \mid 0 < x \leq 1\}$.The infimum of S is 0, since 0 is the largest number that is less than or equal to every element of S.

Properties of Supremum and Infimum:

1. Existence: If a set $S \subset R$ is non-empty and bounded above, then sup$S$ exists. Similarly, if S is non-empty and bounded below, then inf$S$ exists.
2. Uniqueness: The supremum and infimum of a set, if they exist, are unique.
3. Order: For any non-empty set S that is bounded above, sup$S$ is such that:
   $supS \geq s$for all $s \in S$
   For any $\epsilon > 0$, there exists an $s \in S$ such that sup $supS - \epsilon < s \leq supS$
4. Duality: The infimum of a set S is the negative of the supremum of the set $-S$, and vice versa.
   If$T = \{-s \mid s \in S\}$, then $infS = -supT$ and sup $supS = -infT$.

Completeness Property:

The completeness property of the real numbers, also known as the LUB Property, states that every nonempty subset of R that is bounded above has a sup.in R.

If $S \subset R$ is non-empty and bounded above, then $supS \in R$.

Conversely, if $S \subset R$ is non-empty and bounded below, then $infS \in R$.

Importance in Analysis:

1. Existence of Limits:

The completeness property is crucial for the existence of limits. It guarantees that bounded monotone sequences converge.

Example: If $\{an\}$ is a sequence that is bounded and increasing, then $limn \to \infty an = sup\{an \mid n \in N\}$.

2. Interchange of Supremum and Limit:

For a bounded sequence $\{an\}$, $limsup_{n\to\infty} a_n = infsup\{a_k \mid k \geq n\}$ ensures the interchangeability of limit superior and supremum.

3. Real Analysis:

Many theorems in real analysis rely on the completeness of R. For example, the Bolzano-Weierstrass theorem status that every bounded sequence in R has a convergent subsequence.

4. Topology:

The completeness of R underpins the structure of metric spaces, particularly in defining completeness for these spaces.

Examples and Exercises:

Example: Find the supremum and infimum of the set $S = \{x \in R \mid 2 \leq x \leq 5\}$.

Solution: The supremum of S is 5, and the infimum of S is 2.

1.6 Summary:

By the end of an "Introduction to Real Numbers" course, students should have a solid understanding of the real number system, be able to perform and understand various operations with real numbers, and apply these concepts to solve both abstract mathematical problems and practical real-world scenarios. These learning objectives ensure that students build a strong foundation in real numbers, which is essential for further studies in mathematics and related disciplines.

1.7 Keywords:

- Real Number System
- Arithmetic, Order, Algebric properties
- Decimals and factors
- Supremum and Infimum

1.8 Self-Assessment questions:

1. Define a real number.
2. What is the difference between rational and irrational numbers?

3. Provide three examples of irrational numbers.
4. Explain the closure property of real numbers.
5. What is the associative property? Give an example using real numbers.
6. State the distributive property and provide an example.
7. Simplify the expression: $5\sqrt{3} + 2\sqrt{3}$.
8. Evaluate the expression: $(2/3) / (4/5)$.
9. Plot the following numbers on a number line: $-2, 0, 3.5, \sqrt{2}$.

1.9 Case Study:

1. How did the discovery of irrational numbers influence the development of mathematics?
2. In what ways do real numbers appear in everyday life? Provide examples.
3. Discuss the importance of the properties of real numbers in ensuring the consistency of mathematical operations.
4. Create a real-world problem that involves real numbers and solve it, explaining each step.

1.10 References:

- Royden, H., Fitzpatrick, P. (2018). Real Analysis. United Kingdom: Pearson.
- Trench, W. F. (2013). Introduction to Real Analysis. United Kingdom: Prentice Hall/Pearson Education.

UNIT - 2

Continuity and Uniform Continuity

Learning Objectives:

- Understand the Definition of Continuity at a Point
- Recognize Continuous Functions
- Understand and use of Weierstrass's theorem
- Understand the topology and Metric spaces

Structure:

2.1 Understanding Continuity:

Definition: A function $f: D \subseteq R \rightarrow R$ is continuous at a point c in its domain if, for every $\epsilon > 0$, ∃a $\delta > 0$ such that $| f(x) - f(c) | < \epsilon$ whenever $| x - c | < \delta$.

Key Points:

- A function is continuous if it doesn't have any breaks, jumps, or holes in its graph.
- Continuity at a point means that small changes in the input lead to small changes in the output.
- Continuous functions preserve limits: $limx \rightarrow cf(x) = f(c)$.

2.2 Uniform Continuity:

Definition 2.1

A function $f: A \rightarrow R$ defined on a subset A of the real numbers R is said to be uniformly continuous if for every $\epsilon > 0$, ∃a $\delta > 0$:$\forall x, y \in A$,

$| x - y | < \delta \implies | f(x) - f(y) | < \epsilon$.

Key Points:

- Uniform continuity is a stronger condition than continuity. It requires that the choice of δ works uniformly for all points in the domain.
- While continuity focuses on the behavior around individual points, uniform continuity considers the behavior over the entire domain simultaneously.
- Uniformly continuous functions can "control" oscillations and ensure that the function doesn't "vary too much" across the entire domain.

Differences between Continuity and Uniform Continuity:

1. **Existence of δ:**

   For continuity, δ may depend on both ϵ and c.

   For uniform continuity, δ must work for all points simultaneously and doesn't depend on any particular point.

2. **Local vs. Global:**

   Continuity focuses on the behavior of a function at individual points, considering local neighborhoods.

   Uniform continuity considers the behavior of a function across the entire domain, providing a global control on its variations.

3. **Preservation of Cauchy Sequences:**

   Uniform continuity preserves Cauchy sequences. If a function is uniformly continuous on a set, then it maps Cauchy sequences to Cauchy sequences.

Example:

Consider the function $f(x) = 1/x$ defined on the interval $(0, \infty)$.

- $f(x)$ is continuous but not uniformly continuous on $(0, \infty)$.
- While $f(x)$ is continuous at each point in its domain, it exhibits unbounded oscillations as x approaches 0, making it impossible to find a single δ that works uniformly for the entire interval.

2.3 Metric spaces and their topology:

Definition 2.2

A metric space is a pair (X, d) where:

- $X$is a set.
- $d: X \times X \to R$ is a metric on X, satisfying the next properties for all $x, y, z \in X$:

1. **Non-negativity**: $d(x, y) \geq 0$ and $d(x, y) = 0$iff$x = y$.
2. **Symmetry**: $d(x, y) = d(y, x)$.
3. **Triangle Inequality**: $d(x, z) \leq d(x, y) + d(y, z)$.

Examples:

1. **Euclidean Space:**
   - Set: R^n.
   - Metric: The Euclidean distance

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

   for $x = x_1, x_2, \dots x_n$and$y = y_1, y_2, \dots y_n$

2. **Discrete Metric Space:**
   - Set: Any set X.
   - Metric:

$$d(x,y) = \{\begin{matrix} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{matrix}$$

Topology of Metric Spaces:

Open Sets:

- A set U in a metric space X is open if, for every point x in U, there exists a positive real number r such that the open ball $B(x, r)$ is contained in U.
- Open sets are the basic building blocks of the topology of a metric space.

Closed Sets:

- A set F in a metric space X is closed if its complement $X \setminus F$ is open.
- Closed sets contain all their limit points.

Interior, Boundary, and Closure:

- The interior of a set A in X, indicated by $int(A)$, is the largest open set contained in A.
- The boundary of A, indicatedby ∂A, is the set of points in X that are neither in $int(A)$ nor in the complement of A.
- The closure of A, denoted by$\bar{A}$, is the union of A and its boundary.

Convergence:

A function $f: A \to R$ defined on a subset A of the real numbers R is said to be uniformly continuous if for every $\epsilon > 0$, ∃a $\delta > 0$:$\forall x, y \in A$,

$| x - y | < \delta \implies | f(x) - f(y) | < \epsilon$.

Importance in Analysis:

1. **Generalization of Euclidean Spaces**:

   Metric spaces provide a general framework that extends the notion of distance and convergence beyond Euclidean spaces.

2. **Topology and Continuity:**

   The topology induced by a metric space plays a crucial role in defining continuity, open sets, and closed sets, providing a foundation for topological concepts.

3. **Convergence and Completeness:**

   Understanding convergence and completeness in metric spaces is fundamental for analyzing the behavior of sequences and series, as well as for proving the existence and uniqueness of solutions to differential equations.

Example:

Consider the metric space (R, d), where $d(x, y) = | x - y |$ is the standard Euclidean distance function.

- The open interval (a, b) in R is an open set in this metric space.
- The set $[a, b]$ is closed, as its complement $R \setminus [a, b]$ is open.
- The sequence $\{1/n\}$ converges to 0 in (R, d), demonstrating convergence in this metric space.

2.4 Weierstrass's theorem:

Weierstrass's Theorem Statement:

Let f be a continuous function defined on a closed interval $[a, b]$. Then for every $\epsilon > 0$, there exists a polynomial $P(x)$ such that

$sup_{x \in [a,b]} | f(x) - P(x) | < \epsilon$.

Proof

Step 1: Existence of Supremum and Infimum

Since f is continuous on the closed interval $[a,b]$, it is bounded on this interval. By the Extreme Value Theorem, f achieves its supremum and infimum on $[a,b]$. Let M be the supremum and m be the infimumof f on $[a,b]$.

Step 2: Attainment of Maximum

We aim to show that there exists a point c in [a,b] such that $f(c) = M$, the supremum of f on [a,b].

By the definition of supremum, for every positive integer n, there exists a point x_n in [a,b] such that $M - \frac{1}{n} < f(x_n) \leq M$.

Science [a, b] is a closed and bounded interval, by the Bolzano-Weierstrass theorem, the sequence {x_n} has a convergent subsequence $\{x_{n_k}\}$ that converges to some point in [a, b].

Since f is continuous, we have:

$$lim_{k\to\infty} f(x_{n_k}) = f(c)$$

By the squeeze theorem:

$$lim_{k\to\infty} M - \frac{1}{n_k}) \leq f(c) \leq lim_{k\to\infty} M$$

$M \leq f(c) \leq M.$

Thus, $f(c) = M$, and f attains its maximum at c on [a,b].

Step 3: Attainment of Minimum

Similarly, we aim to show that there exists a point d in $[a, b]$such that $f(d) = m$, the infimum of f on $[a, b]$.

Using a similar argument as in Step 2, we can show that there exists a point d in $[a, b]$such that $f(d)=m$, and thus f attains its minimum at d on$[a, b]$.

Conclusion:

Since f attains its maximum at c and its minimum at $d$on $[a, b]$, Weierstrass's theorem is proved.

2.5 Continuity of functions in metric spaces:

In the context of metric spaces, the notion of continuity for functions is defined analogously to that in real analysis. Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f: X \to Y$ be a function.

Definition 2.3

Let (X, d_X) and (Y, d_Y)be metric spaces. A function $f: X \to Y$ is said to be continuous at a point $x_0 \in X$ if for every $\epsilon > 0$, there exists a $\delta > 0$such that for all $x \in X$,

$d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \epsilon.$

Key Points:

- Epsilon-Delta Definition: The definition of continuity in metric spaces mirrors that of real analysis but replaces the absolute value with the metric distance function dY in the codomain.
- Intuition: A function *f* is continuous if small changes in the input *x* result in small changes in the output *f*(*x*), as measured by the metric distance dY.
- Sequential Definition: Alternatively, f is continuous at x_0 if, for every sequence $\{x_n\}$in *X* converging to x_0, the sequence $\{f(x_n)\}$ converges to $f(x_0)$ in *Y*.
- Composition of Continuous Functions: If $f{::}X \rightarrow Y$ and $g{:}Y \rightarrow Z$ are continuous functions between metric spaces, then their composition $g \circ f{:}X \rightarrow Z$ is also continuous.
- Continuity and Open Sets: A function *f* is continuous iff the pre-image of every open set in *Y* is an open set in *X*.

Importance in Analysis:

1. Topology: Continuity is a fundamental concept in topology, as it defines the relationship between the topologies of the domain and codomain of a function.
2. Convergence: Continuous functions preserve convergence, allowing for the analysis of sequences and series in metric spaces.
3. Applications: Continuity plays a crucial role in various fields such as optimization, differential equations, and dynamical systems, where understanding the behavior of functions is essential.

Example:

Considering the function $f{:}R \rightarrow R$ classify by $f(x) = 2x + 1$.

- This function is continuous everywhere on R with respect to the standard Euclidean metric.
- Given any $\epsilon > 0$, if we choose $2\delta = 2\epsilon$, then for any x_0 in R, if $|\ x - x0\ | < \delta$, then $f(x) - f(x0)\ | = |\ 2x + 1 - (2x0 + 1)\ | = 2\ |\ x - x0\ | < 2\delta = \epsilon$.
- Thus, *f* is continuous on R.

2.6 Summary:

Continuity and uniform continuity are fundamental concepts in mathematical analysis that describe how functions behave with respect to small changes in their inputs. Continuity at a point ensures the function's output changes smoothly as the input changes. Uniform continuity is a stronger condition that requires this smooth change to be consistent across the entire domain. These concepts are essential for understanding more advanced topics in calculus and real analysis, including integration, differentiation, and the behavior of sequences and series.

2.7 Keywords:

- Continuous Functions
- Uniform Continuity
- Weierstrass's Theorem
- Metric Space
- Euclidean space
- Subsequence

2.8 Self-Assessment questions:

1. Provide an example of a metric space that is not Euclidean space.
2. Provide an example of a metric space that is not Euclidean space.
3. Prove that a function is continuous iff the pre-image of every open set is open.
4. Give an example of a metric space that is not complete.

2.9 Case Study:

Consider the set of all continuous functions on the interval [0,1], denoted as $C([0,1])$. We define a metric ddd on this space using the supremum norm:

$d(f,g) = \| f - g \|_{\infty} = sup_{x\in[0,1]} \mid f(x) - g(x)$.

Here, f and g are elements of $C([0,1])$.

2.10 References:

- Royden, H., Fitzpatrick, P. (2018). Real Analysis. United Kingdom: Pearson.
- Trench, W. F. (2013). Introduction to Real Analysis. United Kingdom: Prentice Hall/Pearson Education

UNIT-3

Compactness and Connectedness

Learning Objectives:

- Understand the Definition of Compactness
- Explore Properties of Compact Sets
- Understand the Definition of Connectedness
- Explore Properties of Connected Sets

Structure:

3.1 Exploring compact sets:

In the realm of topology and analysis, understanding compact sets is pivotal due to their rich properties and implications in various theorems. Let's delve into the concept of compact sets.

Definition 3.1

A set K in a metric space X is said to be compact if every open cover of K has a finite sub cover. Or for any collection of open sets $\{U_\alpha\}$ such that $\subseteq \cup_\alpha U_\alpha$, there exists a finite subset $\{U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}\}$ such that $K \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$.

Key Properties:

1. Closed and Bounded: In Euclidean spaces, compact sets are closed and bounded. This property is known as the Heine-Borel theorem.
2. Finite Sub cover Property: This is the defining property of compact sets. No matter how finely we cover a compact set with open sets, we can always extract a finite sub cover.
3. Compactness Implies Sequential Compactness: Every sequence in a compact set has a convergent subsequence that converges to a point in the set.
4. Continuous Image of Compact Sets: The image of a compact set under a continuous function is compact. This property is known as the continuity theorem for compact sets.
5. Product of Compact Sets: The Cartesian product of finitely many compact sets is compact. This property is known as the product theorem for compact sets.

Importance in Analysis:

1. Existence of Extrema: Compactness is crucial for proving the existence of maximum and minimum values of continuous functions defined on closed intervals.
2. Convergence: Compact sets facilitate the study of convergence in various contexts, such as sequences, series, and functions.
3. Topology: Compact sets play a central role in topology, serving as a bridge between local and global properties of spaces.
4. Functional Analysis: Compact sets are extensively used in functional analysis, particularly in the study of operator theory, spectral theory, and Banach spaces.

Example:

Consider the closed interval [0,1] in the real line R.

- This set is compact in R according to the Heine-Borel theorem.
- Any open cover of [0,1] can be reduced to a finite sub cover, demonstrating its compactness.

3.2Connectedness in metric spaces:

Connectedness is a fundamental concept in topology that characterizes the "wholeness" or "integrity" of a space. In the context of metric spaces, connectedness plays a crucial role in understanding the structure and behaviour of sets. Let's explore connectedness in metric spaces.

Definition 3.2

If there is no way to split a metric space X into two disjoint non-empty open sets, then the space is said to be linked. The empty set and space X are the only subsets of X that are both open and closed, according to formal definitions of connectedness.

Key Properties:

1. Connected Sets: A subset A of a metric space X is connected if the subspace A is connected with respect to the induced metric topology.
2. Intermediate Value Property: Connectedness is closely related to the intermediate value property. If $f: X \to R$ is a continuous function defined on a connected metric space X, then f takes on all intermediate values between any two given values in its range.
3. Union of Connected Sets: The union of a collection of connected sets that intersect pairwise at least at one point is also connected.

Importance in Analysis:

1. **Topological Characterization:** Connectedness provides a fundamental topological property that helps classify spaces into connected and disconnected ones.
2. **Continuity and Path-connectedness:** Connectedness is intimately linked with the continuity of functions and the existence of paths between points in a space.

3. **Intermediate Value Theorem:** The intermediate value property, a consequence of connectedness, underpins many results in real analysis, including the intermediate value theorem.

Example:

Consider the real line R with the standard Euclidean metric.

- R is a connected metric space. Any attempt to divide R into two disjoint non-empty open sets would fail, as R is an unbroken continuum.
- Any interval (*a*,*b*) in R is also connected. This follows from the fact that any attempt to split the interval into disjoint non-empty open sets would result in one of the sets being empty.

3.3 Discontinuities in functions:

Discontinuities in functions refer to points where the function fails to exhibit continuity. Understanding the nature of discontinuities is crucial in analysis as it provides insights into the behaviour of functions and their limits. Let's explore the different types of discontinuities that can occur in functions defined on metric spaces.

Types of Discontinuities:

1. **Point Discontinuity:** A function $f: X \rightarrow Y$ has a point discontinuity at a point x_0 in the domain if f is not continuous at x_0 but is continuous at all other points in the neighborhood of x_0.
2. **Jump Discontinuity:** A function f:X→Y has a jump discontinuity at a point x_0 in the domain if the one-sided limits$lim_{x \rightarrow x_0^-} f(x)$ and $lim_{\, x \rightarrow x_0^+} f(x)$ exist but are not equal.
3. **Removable Discontinuity:** A function $f: X \rightarrow Y$ has a removable discontinuity at a point x_0 in the domain if the limit$lim_{x \rightarrow x_0^-} f(x)$ exist, but $f(x_0)$ does not equal this limit.
4. **Infinite Discontinuity:** A function f:X→Y has an infinite discontinuity at a point x_0 in the domain if at least one of the one-sided limits $lim_{x \rightarrow x_0^-} f(x)$ or $lim_{\, x \rightarrow x_0^+} f(x)$is infinite.
5. **Oscillatory Discontinuity:** A function $f: X \rightarrow Y$ has an oscillatory discontinuity at a point $x_0$in the domain if f oscillates infinitely near x_0, making it impossible to assign a well-defined limit.

3.4 Monotonic functions:

Monotonic functions are a class of functions that exhibit a consistent trend in their behavior: they either consistently increase or consistently decrease over their entire domain. Understanding monotonic functions is essential in analysis, optimization, and various other fields. Let's explore them further.

Definition 3.3

A function $f: A \rightarrow R$ defined on a set $A \subseteq R$ is said to be:

1. Monotonically Increasing: If for all $x, y \in A$with $x \leq y$, we have $f(x) \leq f(y)$.
2. Monotonically Decreasing: If for all $x, y \in A$with $x \leq y$, we have f $f(x) \geq f(y)$.

Example:

- The identity function$f(x) = x$ on R.
- The exponential function f(x)=e^x on its entire domain.
- The negative identity function$f(x) = -x$ on R.
- The reciprocal function f(x)=1/x on its domain (−∞,0).

3.5 Summary:

Understanding the properties and interplay between connected and compact sets is crucial for many areas of mathematics, including analysis, topology, and geometry. They provide powerful tools for analyzing the structure and behavior of spaces and functions.

3.6 Keywords:

- Connected Sets
- Compact Sets
- Relation between Connected and Compact sets

3.7 Self-Assessment Questions:

1. Explain why the interval [0,1] in R is a connected set.
2. Explain why the interval [0,1] in R is a connected set.
3. Define a compact set in a metric space.
4. Provide an example of a set that R is not compact.
5. State the theorem that every disconnected set in R is not compact.

3.8 Case Study:

The connectedness of [0,1] can be established using the Intermediate Value Theorem. Suppose $f: [0,1] \to R$ is a continuous function. If there exist $a, b \in [0,1]$ such that $f(a) < c < f(b)$, then by IVT (Intermediate Value Theorem), there exists $x \in [a, b]$ such that $f(x) =$ c. This demonstrates that f([0,1])is connected for any continuous function f on [0,1].

3.9 References:

- Royden, H., Fitzpatrick, P. (2018). Real Analysis. United Kingdom: Pearson.
- Trench, W. F. (2013). Introduction to Real Analysis. United Kingdom: Prentice Hall/Pearson Education

Unit - 4

Sequences and Series

Learning Objectives:

- Understand the Definition of a Sequence
- Convergence and Divergence of Sequence
- Understand the Cauchy sequences
- Absolute and conditional convergence

Structure:

4.1 Convergence of sequences:

The convergence of sequences is a fundamental concept in analysis that describes the behaviour of a sequence as its terms approach a specific limit. Understanding convergence is crucial in various areas of mathematics, including calculus, real analysis, and functional analysis. Let's explore the convergence of sequences.

Definition 4.1

A sequence $\{x_n\}$ in a metric space X is said to converge to a limit L if, for every positive real number ϵ, there exists a positive integer N such that for all $n \geq N$, the distance between x_n and L is less than ϵ. Symbolically, this is expressed as:

$$lim_{n\to\infty} x_n = L$$

Key Concepts:

1. Limit: The limit L is the value that the terms of the sequence approach as n tends to infinity.
2. Convergence Criterion: When all terms in a series are within ϵ distance of the limit at any point beyond which the sequence exists, for any arbitrarily small positive integer ϵ, the sequence is said to be convergent.
3. Divergence: If a sequence does not converge, it is said to diverge. Divergence can occur in various forms, such as unboundedness, oscillation, or failure to approach any specific value.
4. Limit Notation: Convergence is often denoted using the limit notation $lim_{n\to\infty} x_n = L$, where L is the limit of the sequence.

Example:

This sequence converges to 0 as n tends to infinity, as for any $\epsilon > 0$, we can choose N such that $1/N < \epsilon$ for all $n \geq N$.

4.2 Cauchy sequences:

Cauchy sequences are an important concept in real analysis and the theory of metric spaces. They represent a specific type of sequence where the terms become arbitrarily close to each other as the sequence progresses. Let's explore Cauchy sequences further.

Definition 4.2

A sequence $\{x_n\}$ in a metric space X is called a Cauchy sequence if, "for every positive real number ϵ, there exists a positive integer N such that for all $m,n \geq N$", the distance between x_m and x_n is less than ϵ. Symbolically, this is expressed as:

for all $\epsilon > 0, \exists N \in N$ such that $d(x_m, x_n) < \epsilon$ for all $m, n \geq N$.

Example:

Consider the sequence $\{x_n\}=\{\frac{1}{n}\}$ in the real numbers.

- This sequence is a Cauchy sequence because for any $\epsilon>0$, we can choose N such that $\frac{1}{m} - \frac{1}{n} < \epsilon$ for all $m, n \geq N$.
- Alternatively, consider the sequence $\{y_n\} = \{1 + \frac{1}{2^n}\}$. This sequence is also a Cauchy sequence because the terms approach 1 as n tends to infinity.

4.3 Upper and Lower limits:

Upper and lower limits, also known as the supremum and infimum, respectively, play a crucial role in analyzing the behaviour of sequences and sets, particularly in real analysis and the theory of metric spaces. Let's explore upper and lower limits further.

Upper Limit (Supremum):

Any real number that is smaller than or equal to every element in a set 'S' of real numbers is its supremum, or upper bound. It's represented by sup(S).

Formally:

sup(S)=smallest x such that $x \geq s$ for all $s \in S$

Lower Limit (Infimum):

For each set S of real numbers, the greatest real number less than or equal to all of S's elements is its lower limit, also known as its infimum. It's represented by inf(S).

Formally:

inf(S)=largest x such that $x \leq s$ for all $s \in S$

Example:

Consider the set $S = \{1 - \frac{1}{n} \mid n \in N\}$ of real numbers.

- The supremum of *S* is sup(*S*)=1 because 1 is the smallest real number greater than or equal to all elements of *S*.
- The infimum of *S* is inf(*S*)=0 because 0 is the largest real number less than or equal to all elements of *S*.

4.4 Cauchy's general Principle of convergence:

Cauchy's General Principle of Convergence, also known simply as Cauchy's Convergence Criterion, is a fundamental concept in real analysis. It provides a criterion for determining when a sequence converges based on the sequence itself, without reference to a specific limit. Let's explore Cauchy's Convergence Criterion further.

Definition 4.3

Cauchy's Convergence Criterion states that a sequence $\{x_n\}$ in a metric space *X* converges if and only if, for every positive real number ϵ, there exists a positive integer *N* such that for all $m,n \geq N$, the distance between $x_m$and $x_n$is less than ϵ. Symbolically:

The sequence $\{x_n\}$ converges $\Leftrightarrow \forall \epsilon > 0, \exists N \in N$ such that $d(x_m,x_n)<\epsilon$ for all $m, n \geq N$

Example

Consider the sequence $\{x_m\}=\{1/n\}$ in the real numbers.

This sequence satisfies Cauchy's Convergence Criterion because for any $\epsilon>0$, we can choose N such that $1/m-1/n<\epsilon$ for all $m,n \geq N$.

Squeeze Theorem (or Sandwich Theorem):

Theorem:

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If there exists an integer Nsuch that for all $n \geq N$, $a_n \leq b_n \leq c_n$, and if $\lim_{n\to\infty} an= \lim_{n\to\infty} c_n=L$ then $\lim_{n\to\infty} b_n= L$.

Proof:

By the definition of the limit, for every $\epsilon>0$, there exists a positive integer N_1such that for all $n\geq N_1$,

$|a_n-L|<\epsilon$.

This means $L-\epsilon<a_n<L+\epsilon$.

Similarly, for every $\epsilon> 0$, there exists a positive integer N_2 such that for all $n\geq N_2$,

$|c_n-L|<\epsilon$.

This means $L-\epsilon<c_n<L+\epsilon$.

Let $N_0=\max(N,N_1,N_2)$. For all $n\geq N_0$,

$L-\epsilon<a_n\leq b_n\leq c_n<L+\epsilon$.

Therefore, for all $n\geq N_0$,

$L-\epsilon<b_n<L+\epsilon$,

Which implies $|b_n-L|<\epsilon$.

4.5 Summary:

Understanding sequences and series, their properties, and convergence criteria are crucial for advanced studies in mathematics and its applications in science and engineering.

4.6 Keywords:

- Sequences
- Series
- Convergence Tests

4.7 Self-Assessment Questions:

1. What does it mean for a sequence $\{a_n\}$ to converge to a limit L? Provide the formal definition.

2. Determine whether the sequence $\{b_n\}=1/n$ converges or diverges. If it converges, find its limit.

3. Is every bounded sequence convergent? Provide a justification for your answer.

4. Given two convergent sequences $\{a_n\}$ and $\{b_n\}$ with limits A and B respectively, what is the limit of the sequence $\{c_n\}$ where $c_n=a_n+b_n$?

5. Use the Squeeze Theorem to determine the limit of the sequence {sinn/n}.

4.8 Case Study:

A meteorologist is studying the average monthly temperatures over several years to predict future climate patterns. They have collected temperature data T_n for a specific location over n months. The goal is to determine if the average temperature sequence converges, which would imply a stable long-term climate trend, or if it shows signs of divergence, indicating possible climate change.

Question:

Suppose the temperature data for the past 60 months (5 years) is as follows:

T={30.5,31.0,30.7,30.9,31.2,30.8,30.6,31.0,31.1,30.9,…,31.0}

To analyze the trend, we construct the sequence of the average temperature $\{A_n\}$, where A_n is the average temperature over the first n months.

$$A_n = \frac{1}{n}\sum_{i=1}^{n} T_i$$

4.9 References:

- Royden, H., Fitzpatrick, P. (2018). Real Analysis. United Kingdom: Pearson.
- Trench, W. F. (2013). Introduction to Real Analysis. United Kingdom: Prentice Hall/Pearson Education

Unit – 5

Bounded Sequence

Learning Objective

- Understand the Definition and Concept of Bounded.
- Identify whether a given sequence is bounded by examining its terms.
- Understand how the sum, difference, product, and quotient of bounded sequences are bounded.

Structure

5.1 Bounded Sequence

A sequence (a_n) is a function a:N→R where a(n)=an. A sequence can be described as an ordered list of real numbers indexed by natural numbers.

A sequence (a_n) is said to be **bounded** if there exists a real number $M > 0$ such that for all n ∈ N , the absolute value of the sequence's terms is less than or equal to M. Formally, "(a_n) is bounded if there exists M>0" such that:

$$|\, an \,| \leq \; M \text{for all } n \; \in \; N.$$

In other words, both the sequence and its negative are bounded above. There are two key types of bounded sequences:

1. **Bounded Above**: "A sequence (a_n) is bounded above if there exists a real number B such that $a_n \leq$ B".
2. **Bounded Below**: "A sequence (a_n) is bounded below if there exists a real number C such that $a_n \geq$ C for all n ∈ N".

A sequence that is both bounded above and bounded below is simply referred to as bounded.

examples of sequences that are bounded above and bounded below:

1. Bounded Above Sequence:

$$a_n = \frac{n}{n+1} \; for \; n \geq 1$$

Solution: This sequence $(\frac{n}{n+1})_{n=1}^{\infty}$ is bounded above because for all $n \geq 1$,. Specifically, $\frac{n}{n+1}$ approaches 1 as n increases, but it never reaches or exceeds 1.

Therefore, 1 is an upper bound for this sequence.

2. Bounded Below Sequence:

$$b_n = (-1)^n$$

Solution: The sequence $(-1)^n$alternates between -1 and 1 as n varies.

Hence, $((-1)^n)_{n=1}^{\infty}$ is bounded below by -1 because for every n, $(-1)^n = -1$ when n is odd. Therefore, -1 is a lower bound for this sequence.

Theorem 1: Every convergent sequence is bounded.

Proof: Let (a_n) be a sequence that converges to a limit L. By the definition of convergence, for every $\epsilon>0$, there exists a natural number N such that for all $n \geq N$,

$$|a_n - L| < \epsilon.$$

Choose $\epsilon=1$. Then there exists a natural number N such that for all $n \geq N$,

$$|a_n - L| < 1.$$

This implies that for all $n \geq N$,

$$-1 < a_n - L < 1.$$

Adding L to all parts of the inequality, we get:

$|L-1| < a_n < |L+1|$.

Therefore, for $n \geq N$,

$|an| \leq \max(|L-1|,|L+1|)$..

Let $M_1=\max(|L-1|,|L+1|)$. This M_1 bounds all terms of the sequence a_n forn $\geq$ N.

For n < N, the terms $a_1,a_2,\ldots,a_{N-1}$ are a finite number of terms. Let

$M_2=\max(|a_1|,|a_2|,\ldots,|a_{N-1}|)$.

Now, define

$M=\max(M_1,M_2)$.

Since M_1 bounds all terms a_n for $n \geq N$ and M_2 bounds the finite number of terms $a_1,a_2,\ldots,a_{N-1}$, M will be an upper bound for the absolute values of all terms of the sequence (a_n).

Thus, for all $n \in N$,

$|an| \leq M$.

Therefore, the sequence (a_n) is bounded.

Hence we can say that every sequence that converges has a limit. This is because convergence indicates that the sequence's terms approach the limit arbitrarily, indicating that the existence of a bound for each term in the sequence is required.

Example 1: Consider the sequence

$$a_n = \frac{n+1}{n} \; for \; n \geq 1$$

Solution: This sequence $(\frac{n+1}{n})_{n=1}^{\infty}$ is bounded above because for all $n \geq 1$, $\frac{n+1}{n} = 1 + \frac{1}{n} < 2$. Therefore, 2 is an upper bound for this sequence.

5.2 Convergent Sequences

Convergence of a real number sequence (sn) to real number s is defined as follows:

$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N}, \quad \text{such that } n > N \text{ implies } |s_n - s| < \varepsilon. \qquad (1)$$

If this is the case, we write s = limn→∞ sn or sn → s, denoting that (sn) is a convergence sequence with s as its limit. We refer to (sn) as a divergent sequence if it does not converge. First, we demonstrate that a single sequence (sn) cannot have two distinct bounds. Assume that sn → t and sn → s. Assume ε > 0. Consequently, ε / 2 > 0. Since sn → s, by definition, for n > N1, |sn − s| < ε / 2 exists in N1. If n > N2, then by definition, N2 ∈ N such that |sn − t| < ε / 2 because sn → t. Because the N originating from the two bounds might not be the same, we utilize N1 and N2 in these two assertions. N = max{N1, N2}, please. In the event that n > N,

$$|s - t| \leq |s_n - s| + |s_n - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

For any ε > 0, |s−t| < ε now holds. After that, we determine that |s−t| = 0 (because if |s−t| > 0, we would have a contradiction, thus we would choose ε = |s − t|). Since s = t, the uniqueness is maintained.

Example 2: We have $1 / n \to 0$.

Proof: Let $\varepsilon > 0$. By Archimedean property, there is $N \in N$ such that $1 / N < \varepsilon$. If $n > N$, then

$$|\frac{1}{n} - 0| = \frac{1}{n} < \frac{1}{N} < \varepsilon.$$

1. Limit of a Sequence

If (a_n) is a sequence of real numbers, we say that the sequence converges to the limit L if for every $\epsilon > 0$, there exists a natural number N such that for all $n \geq N$,

$|a_n - L| < \epsilon$.

We write this as:

$\lim_{n\to\infty} a_n = L$.

2. Limit of a Function

If f(x) is a function of a real variable, we say that the limit of f(x) as x approaches c is L, written as $\lim_{x\to c} f(x) = L$, if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all x with $0 < |x-c| < \delta$,

$|f(x) - L| < \epsilon$.

Limit Laws

If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then the following hold:

- **Sum Rule**: $\lim_{n\to\infty} (a_n + b_n) = A + B$
- **Difference Rule**: $\lim_{n\to\infty} (a_n - b_n) = A - B$
- **Product Rule**: $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
- **Quotient Rule**: $\lim_{n\to\infty}(a_n / b_n) = A / B$, provided $B \neq 0$
- **Scalar Multiplication**: $\lim_{n\to\infty}(c \cdot a_n) = c \cdot A$, for any real number c

5.3 Summary

A bounded sequence is a sequence of numbers $\{a_n\}$ where there exists a real number M such that $|a_n| \leq M$ for all n. This means the sequence's terms do not grow indefinitely in either direction. Every convergent sequence is bounded. The sum, difference, and product of bounded sequences are also bounded. Identifying and proving boundedness involves finding appropriate bounds and applying these concepts to various mathematical problems.

5.4 Keywords

- Supremum
- Infimum
- Limit Superior (lim sup)
- Limit Inferior (liminf)
- Absolute Value
- Upper Bound
- Lower Bound

5.5 Self Assessment

1. What is the formal definition of a bounded sequence, and how can you determine if the sequence $a_n = \sin(\mathrm{n})$ is bounded?
2. Prove that the sequence $a_n = \frac{(-1)^{\mathrm{n}}}{\mathrm{n}}$ is bounded.
3. Explain why every convergent sequence is bounded and provide an example to illustrate this property.
4. Compare and contrast a bounded sequence and an unbounded sequence, giving an example of each and explaining the key differences.
5. How can the concept of bounded sequences be applied in real-world scenarios or in other areas of mathematics, such as in solving differential equations or in the context of functional analysis?

5.6 Case Study

A scientist is modeling the population of a species in a confined habitat. They propose that the population at time n, P_n , follows the sequence $P_n = \frac{1000}{1+0.01n}$.

Questions:

1. Determine if the population sequence $\{P_n\}$ is bounded.
2. What are the implications of the sequence being bounded in terms of population control?

3. How does the bounded nature of the sequence influence long-term predictions for the species' population?

5.7 References

- Royden, H., Fitzpatrick, P. (2018). Real Analysis. United Kingdom: Pearson.
- Trench, W. F. (2013). Introduction to Real Analysis. United Kingdom: Prentice Hall/Pearson Education

Unit - 6
Monotonic sequence

Learning Objective

- Understand the Definition and Concept of Monotonic Sequences.
- Identify whether a given sequence is increasing, decreasing, non-increasing, or non-decreasing by examining its terms.
- Understand that every bounded monotonic sequence converges and be able to apply this theorem to various sequences.

Structure

6.1 Introduction to Monotonic Sequence

Monotonicity and boundedness are two very strong characteristics in the context of testing convergence and divergence of real sequences. We have a very important theorem, namely, the monotone convergence theorem, important in the sense of its usefulness and application, which not only ensures us that a bounded monotonic sequence of real numbers is always convergent, but also unambiguously points out the fact that such a sequence always converges to its lub (least upper bound or supremum)/glb (greatest lower bound or infimum) depending on the monotonicity nature of the particular sequence. So, it is considered as a strong tool of analysis for handling the problem of testing convergence/divergence of monotonic sequences.

In this discourse, we will mainly discuss about monotonic sequences and the very useful monotonic convergence theorem in detail. We will also deal with some particular problems where monotone convergence theorem can be applied very efficiently.

6.2 Monotonic sequence

A monotonic sequence is a sequence of numbers that either consistently increases or consistently decreases as you move through the sequence.

1. Monotonic Increasing Sequence: This occurs when each term in the sequence is greater than or equal to the previous one. Formally, for a sequence a_n:

$$a_1 \leq a_2 \leq a_3 \leq \cdots \ldots \ldots$$

Example 1:

Monotonic Increasing Sequence: $1,3,5,7,9, \ldots \ldots \ldots$

In other words,

A sequence a_n is monotonically increasing if $a_{n+1} \geq a_n$ for all n.

Example 2: $1, 2, 2, 3, 4, 5$ (each term is greater than or equal to the previous one).

Strictly Increasing Sequence:

A sequence a_n is strictly increasing if $a_{n+1} > a_n$ for all n.

Example 3: 1,2,3,4,5,6 (each term is greater than the previous one).

2. Monotonic Decreasing Sequence: This occurs when each term in the sequence is less than or equal to the previous one. Formally, for a sequence a_n:

$$a_1 \geq a_2 \geq a_3 \geq \cdots \ldots \ldots$$

Example 4:
Monotonic Decreasing Sequence: $100, 98, 95, 91, 86, \ldots \ldots \ldots$
In other words,

A sequence a_n is monotonically decreasing if $a_{n+1} \leq a_n$ for all n.

Example 5: 5,4,4,3,2,1 (each term is less than or equal to the previous one).

Strictly Decreasing Sequence:

A sequence (an)(a_n)(an) is strictly decreasing if $a_{n+1} < a_n$for all n.

Example 6: 6,5,4,3,2,1 (each term is less than the previous one).

Properties of Monotonic Sequences

- **Boundedness:** A monotonic sequence that is bounded above (for increasing sequences) or below (for decreasing sequences) will converge to a limit.
- **Convergence:** Monotonic sequences are particularly important in the study of limits and convergence. If a monotonic sequence is bounded, it is guaranteed to converge to a finite limit.

Example 7: Determine the sequence $a_n = \frac{1}{n},$ is monotonic.

Solution: To determine if the sequence $a_n = \frac{1}{n}$, is monotonic, we need to check if it is either monotonically increasing or decreasing. Here's a detailed step-by-step solution:

Step-by-Step Solution

1. **Comparison of Terms:**

   To determine if the sequence is monotonic, compare consecutive terms a_n and a_{n+1}:

$$a_n = \frac{1}{n}$$
$$a_{n+1} = \frac{1}{n+1}$$

2. **Determine the Relationship:**
   We need to find the relationship between a_n and a_{n+1}. Specifically, we need to determine if $a_{n+1} < a_n$ or $a_{n+1} \leq a_n$.

   $a_{n+1} < a_n$ if and only if $\frac{1}{n+1} \leq \frac{1}{n}$

3. **Simplify the Inequality:**

   To simplify the inequality $\frac{1}{n+1} \leq \frac{1}{n}$:
   $$\frac{1}{n+1} \leq \frac{1}{n}$$
   Since n and $n+1$ are positive integers and $n < n+1$, taking the reciprocal of both sides (which reverses the inequality for positive numbers) gives:
   $$\frac{1}{n+1} < \frac{1}{n}$$

   This inequality is true for all positive integers n.
   Hence, the inequality $\frac{1}{n+1} < \frac{1}{n}$ holds for all n. Therefore, the sequence $a_n = \frac{1}{n}$ is monotonically decreasing.

6.3 The monotone Convergence Theorem

Statement: A real monotonic sequence is convergent if and only if it is bounded. Further, if X=$\langle x_n \rangle$ is bounded and monotonic

i) Increasing, then, lim$\langle x_n \rangle$ = sup$\{x_n : n \in N\}$

ii) Decreasing, then, lim$\langle x_n \rangle$ = inf$\{ x_n : n \in N\}$

Proof:

We have already proved that a convergent sequence is always bounded. So, one part of the proof is already done.

Conversely, let, X=$\langle x_n \rangle$ be bounded. Then there may arise two cases: (i) $\langle x_n \rangle$ is monotonically increasing or (ii) X=$\langle x_n \rangle$ is monotonically decreasing.

Case (i): Let $\langle x_n \rangle$ be monotonically increasing.

Now, X=$\langle x_n \rangle$ is bounded $\Rightarrow \exists$ M$\in$R s.t.$x_n \leq$ M $\forall$ n$\in$M .

Then, by the completeness Property, the supremum x*= sup$\{ x_n : = n \in N\}$ must belong to R.

We now show that lim$\langle x_n \rangle$=x*=sup$\{x_n : = n \in N\}$

Let 0>ε be arbitrary. Then,

$$\varepsilon > 0,\ x^* = \sup\{x_n : n \in \mathrm{N}\} \Rightarrow x^* - \varepsilon \text{ is not an upper bound of } X = \langle x_n \rangle$$
$$\Rightarrow \exists \text{at least one } n_0 \in \mathrm{N} \text{ s.t. } x^* - \varepsilon < x_{n_0}$$

Since, $X=<x_n>$ is monotonic increasing, so

$x^* - \varepsilon < x_n, \forall n \geq n_0$.

Further, we have,

$$x^* - \varepsilon < x_{n_0} \leq x_n \leq x^* < x^* + \varepsilon, \quad \forall n \geq n_0$$
$$\Rightarrow x^* - \varepsilon < x_n < x^* + \varepsilon, \quad \forall n \geq n_0$$
$$\Rightarrow |x_n - x^*| < \varepsilon, \quad \forall n \geq n_0$$
$$\Rightarrow \lim (x_n) = x^* = \sup\{x_n : n \in \mathrm{N}\}$$

Case (ii): Let $<x_n>$ be monotonically decreasing so that $x_n \geq x_{n+1}$, $\forall$ $n \in$N.

Now

$X = \langle x_n \rangle$ is bounded $\Rightarrow \exists b \in R$ *s.t.* $x_n \geq b, \forall n \in \mathrm{N}$.

Then, by the completeness Property, the supremum $x^* = \sup\{ x_n : = n \in \mathrm{N}\}$ must belong to R.

We now show that

$\lim \langle x_n \rangle = x^* = \inf\{x_n : n \in \mathrm{N}\}$.

Let $\varepsilon>0$ be arbitrary. Then, we have,

$$\varepsilon > 0,\ x^* = \inf\{x_n : n \in \mathrm{N}\} \Rightarrow x^* + \varepsilon \textit{ is not a lower bound of } X = \langle x_n \rangle$$
$$\Rightarrow \exists \textit{at least one } n_0 \in \mathrm{N} \textit{ s.t. } x^* + \varepsilon > x_{n_0}$$

Since, $X=<x_n>$ is monotonic decreasing, so,

$x^* + \varepsilon > x_{n_0} \geq x_n, \forall n \geq n_0$.

Further, we have,

$$x^* - \varepsilon \leq x_n - \varepsilon < x_n \leq x_{n_0} \leq x^* + \varepsilon, \quad \forall n \geq n_0$$
$$\Rightarrow x^* - \varepsilon < x_n < x^* + \varepsilon, \quad \forall n \geq n_0$$
$$\Rightarrow |x_n - x^*| < \varepsilon, \quad \forall n \geq n_0$$
$$\Rightarrow \lim (x_n) = x^* = \inf\{x_n : n \in \mathrm{N}\}$$

Example 7:

$$\lim\left(\frac{1}{\sqrt{n}}\right) = 0$$

Solution

Here, $n \geq 1, \forall n \in \mathrm{N} \Rightarrow \sqrt{n} \geq 1$ *and hence* $\frac{1}{\sqrt{n}} \leq 1$. So, $\left\langle \frac{1}{\sqrt{n}} \right\rangle$ is bounded.

Further,

$$n+1 > n, \quad \forall n \in \mathrm{N} \Rightarrow \sqrt{n+1} \geq \sqrt{n}$$
$$\Rightarrow \frac{1}{\sqrt{n}} \geq \frac{1}{\sqrt{n+1}}$$

So, it follows that the sequence $X = \left\langle \frac{1}{\sqrt{n}} \right\rangle$ is a monotonic decreasing sequence. Hence, it is a convergent sequence.

Let $\lim X = \lim\left(\frac{1}{\sqrt{n}} \right)$. Then by limit theorem, we have,

$$\lim X.X = \lim\left(\frac{1}{\sqrt{n}}.\frac{1}{\sqrt{n}} \right) = \lim\left(\frac{1}{n}. \right) = 0$$
$$\Rightarrow \lim X.\lim X = [\lim X]^2 = 0$$
$$\Rightarrow \lim X = \lim\left\langle \frac{1}{\sqrt{n}} \right\rangle = 0$$

6.4 Summary

A monotonic sequence is a sequence of numbers that is either entirely non-increasing or non-decreasing. Specifically, an increasing sequence has $a_{n+1} \geq a_n$ for all n, and a decreasing sequence has $a_{n+1} \leq a_n$ for all n. Monotonic sequences exhibit consistent behavior, either rising or falling. A key property is that every bounded monotonic sequence converges to a limit, known as the Monotone Convergence Theorem. Identifying and proving monotonicity involves analyzing the terms' relationships. Monotonic sequences have applications in calculus, analysis, and real-world modeling, providing a foundation for understanding more complex mathematical concepts.

6.5 Keywords

- Monotonic
- Sequence
- Increasing
- Decreasing
- Non-increasing
- Non-decreasing

- Monotone Convergence Theorem
- Bounded
- Unbounded

6.6 Self Assessment

1. What is the formal definition of a monotonic sequence, and how can you determine if the sequence $a_n = n + \frac{1}{n}$ is monotonic?
2. Prove that the sequence $a_n = \frac{1}{n}$ is monotonic and determine whether it is increasing or decreasing.
3. State and explain the Monotone Convergence Theorem. Provide an example of a bounded monotonic sequence and demonstrate how this theorem applies to it.
4. Discuss the implications of a sequence being monotonic in a real-world context, such as predicting trends in financial markets or modeling population growth. Provide a specific example to illustrate your point.
5. Given a monotonic sequence $\{ a_n \}$ that is bounded above, explain why it must converge. How would you go about finding its limit? Use the sequence $a_n = 1 - \frac{1}{n}$ as an example in your explanation.

6.7 Case Study

An investment firm analyzes the performance of a stock over n days, where the daily closing price P_n follows the sequence $P_n = 100 - \frac{1}{n}$.

Questions:

1. Determine if the sequence $\{P_n\}$ is monotonic and classify its monotonicity.
2. Discuss how the monotonic nature of the sequence influences the firm's decision-making process in terms of predicting the stock's future performance.
3. How might the firm use mathematical tools related to monotonic sequences to enhance its investment strategies?

6.8 References

- Royden, H., Fitzpatrick, P. (2018). Real Analysis. United Kingdom: Pearson.
- Trench, W. F. (2013). Introduction to Real Analysis. United Kingdom: Prentice Hall/Pearson Education

Unit -7
Infinite Series

Learning Objective

- Understand the difference between absolute convergence (converges absolutely) and conditional convergence (converges conditionally but not absolutely).
- Understand the implications of series rearrangement and conditions under which it preserves convergence.
- Learn proof techniques related to series convergence and divergence.

Structure

7.1 Introduction to infinite Series

Real numbers are defined as a sequence of function f: N→R, where N is a set of natural numbers and R is a set of real numbers. $(f_1,f_2,...,f_n,...)$ or $\langle f_n \rangle$ can be used to define a series. Take a series, for instance.

Convergent sequence: If there is a positive integer m that depends on ε and for which there is a given ε>0, then the sequence $\langle f_n \rangle$ converges to a number l.

$$|u_n - l| < \varepsilon \ \forall n \geq m.$$

Then l is called the limit of the given sequence and we can write

$$\lim_{n\to\infty} u_n = l \text{ or } u_n \to l$$

7.2 Definition of an Infinite Series

An expression of the form

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

is known as the infinite series of real numbers, where each u_n is a real number. It is denoted by An infinite series is one example. An infinite series' convergence Think of an endless chain. Let's clarify, and so on. The series so created is then referred to as the sequence of partial sums (S.O.P.S.) of the specified series.

Convergent series: A series

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots = \sum_{n=1}^{\infty} u_n$$

converges if $\langle S_n \rangle$ of its partial sums converges i.e. if

$$\lim_{n\to\infty} S_n \text{exists.}$$

exists. Also

$$\lim_{n\to\infty} S_n = S$$

if then S is called as the sum of the given series .

Divergent series: A series

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots = \sum_{n=1}^{\infty} u_n$$

diverges if $\langle S_n \rangle$ of its partial sums diverges i.e. if

$$\lim_{n\to\infty} S_n = +\infty \ or - \infty.$$

Example 1:

Show that the Geometric series$\sum_{n=1}^{\infty} r^{n-1} = 1 + r + r^2 + r^3 + \ldots$,where $r > 0$, is convergent if $r < 1$ and diverges if $r \geq 1$.

Solution:

Let

$S_1 = 1$, $S_2 = 1 + r$, $S_3 = 1 + r + r^2$, ...,

$S_n = 1 + r + r^2 + \cdots + r^{n-1}$

Case 1: r<1

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-r^n}{1-r} = \frac{1}{1-r} - \lim_{n\to\infty} \frac{r^n}{1-r}$$

$$= \frac{1}{1-r} \quad (\text{As } \lim_{n\to\infty} r^n = 0 \text{ if } |r| < 1)$$

Since

$$\lim_{n\to\infty} S_n$$

is finite so $<S_n>$ converges and hence the given series converges.

Case2: r>1

Consider

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{r^n-1}{r-1} = \lim_{n\to\infty} \frac{r^n}{1-r} - \frac{1}{r-1}$$

Since $<S_n>$ diverges and hence the given series diverges.

Case2: r=1

Consider

$S_n = 1 + r + r^2 + \cdots + r^{n-1}$

$$= 1 + 1 + 1 + 1 + \cdots .. + 1 = n \Rightarrow \lim_{n\to\infty} S_n = \infty$$

Since $<S_n>$ diverges and hence the given series diverges.

Positive term series

A positive term series is an infinite series with all positive terms.

p-series: An infinite series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots (p > 0)$$

is called p-series. If p>1 converges if and diverges if p<1.

For Examples

1.$\sum_{n=1}^{\infty}\frac{1}{n^3}=\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+\cdots$ converges (As $p=3>1$)

2.$\sum_{n=1}^{\infty}\frac{1}{n^{5/2}}=\frac{1}{1^{5/2}}+\frac{1}{2^{5/2}}+\frac{1}{3^{5/2}}+\cdots$ converges $\left(\text{As } p=\frac{5}{2}>1\right)$

3.$\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}=\frac{1}{1^{1/2}}+\frac{1}{2^{1/2}}+\frac{1}{3^{1/2}}+\cdots$ converges $\left(\text{As } p=\frac{1}{2}<1\right)$

7.3 Necessary condition for convergence:

If an infinite series $\sum_{n=1}^{\infty}u_n$is convergent then $\lim_{n\to\infty}u_n$. However , converse need not be true.

Proof:

Consider <S_n> of partial sums of the series $\sum_{n=1}^{\infty}u_n$.

As we know that

$$S_n=u_1+u_2+u_3+\cdots\cdots\cdots+u_n$$

$$=u_1+u_2+u_3+\cdots+u_{n-1}+u_n$$

$$\Rightarrow S_{n-1}=u_1+u_2+u_3+\cdots\cdots\cdots+u_{n-1}$$

Now

$$S_n-S_{n-1}=u_n$$

Taking limit $n\to\infty$, we get

$$\lim_{n\to\infty}(S_n-S_{n-1})=\lim_{n\to\infty}u_n$$

$$\Rightarrow \lim_{n\to\infty}S_n-\lim_{n\to\infty}S_{n-1}=\lim_{n\to\infty}u_n \;\ldots\ldots\ldots..(1)$$

As $\sum_{n=1}^{\infty}u_n$ is convergent sequence so <S_n> is also convergent.

Let $\lim_{n\to\infty}S_n=l$, then $\lim_{n\to\infty}S_{n-1}=l$

Substituting these values in (1), we get $\lim_{n\to\infty}u_n$=0.

To show that converse may not hold, let us consider the series

$$\sum_{n=1}^{\infty}u_n=\sum_{n=1}^{\infty}\frac{1}{n}.$$

Here

$$\lim_{n\to\infty}u_n=\lim_{n\to\infty}\frac{1}{n}=0$$

But $\sum_{n=1}^{\infty}\frac{1}{n}$ is a divergent series (As p=1).

Corollary : If $\lim_{n\to\infty}u_n\neq 0$, then $\sum_{n=1}^{\infty}u_n$ cannot converge.

Example 2: Test the convergence of the series $\sum_{n=1}^{\infty}cos\frac{1}{n}$.

Solution:

Here

$$u_n = cos\frac{1}{n} \Rightarrow \lim_{n\to\infty} u_n = \lim_{n\to\infty} cos\frac{1}{n} = 1 \neq 0$$

Hence the series is not convergent.

Example 3: Test the convergence of the series

$$\sum_{n=1}^{\infty} \sqrt{\frac{n}{n+1}}$$

Solution:

Here

$$u_n = \sqrt{\frac{n}{n+1}} \Rightarrow \lim_{n\to\infty} u_n = \lim_{n\to\infty} \sqrt{\frac{n}{n+1}}$$

$$\Rightarrow \lim_{n\to\infty} u_n = \lim_{n\to\infty} \sqrt{\frac{1}{1+\frac{1}{n}}} = 1 \neq 0$$

Hence the series is not convergent.

7.4 Summary

Infinite series are sums of an infinite sequence of terms $\sum_{n=1}^{\infty} a_n$. Understanding centers on convergence criteria like the nth-term test, geometric series test, and ratio test. Series may converge absolutely, conditionally, or diverge. Properties include linearity and the ability to rearrange under specific conditions. Applications span calculus (Taylor series), physics (Fourier series), finance (compound interest), and beyond. Techniques involve manipulating series and utilizing computational tools for analysis. Advanced topics include power series and complex analysis applications. Mastery involves applying convergence tests, understanding series properties, and using them effectively in mathematical modeling and problem-solving.

7.5 Keywords

- Infinite Series
- Convergence
- Divergence
- Summation
- Partial Sum
- Series
- Power Series
- Convergence Tests

7.6 Self Assessment

1. Determine the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Identify the convergence test applicable and justify your conclusion.
2. Given the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, find the sum of the series if it converges. Explain any conditions under which the series converges.
3. In physics, the displacement of a vibrating string is modeled by the series $\sum_{n=1}^{\infty} \frac{1}{n^2} \sin(\frac{n\pi x}{L})$where x is the position along the string and L is its length. Discuss the convergence of this series and its implications for understanding the vibration behavior of the string.
4. Expand e^xinto its Taylor series representation and determine the radius of convergence of the series. Explain how the Taylor series can be used to approximate e^x for different values of x.
5. A bank offers an investment plan where 1000 dollars are invested today, and each subsequent year, the amount grows by 5% compounded annually. Represent the future value of the investment using an infinite series and calculate the total amount accumulated after 10 years. Interpret your result in terms of the convergence of the series and the growth of the investment.

7.7 Case Study

A mathematician studies the behavior of the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$.

Questions:

1. Determine the convergence of the series using appropriate tests and justify your conclusion.
2. Explain whether the series converges absolutely, conditionally, or diverges.
3. How might the mathematician extend this analysis to study similar alternating series and their properties?

7.8 References

- Royden, H., Fitzpatrick, P. (2018). Real Analysis. United Kingdom: Pearson.
- Trench, W. F. (2013). Introduction to Real Analysis. United Kingdom: Prentice Hall/Pearson Education

Unit – 8

Alternating Series

Learning Objective

- Understand when an alternating series converges conditionally and when it converges absolutely.
- Explore methods for summing alternating series, including partial sums and convergence properties.
- Utilize alternating series in mathematical models, such as Taylor series expansions and power series representations.

Structure

8.1 Introduction to Alternating Series

A series in which the words alternate between positive and negative signs is known as an alternating series. Alternating series are important in calculus and mathematical analysis because they help determine approximate values of sums and comprehend convergence features.

Alternating Series

An alternating series is a series where the terms alternate in sign. Mathematically, an alternating series can be written in the form:

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n$$

or

$$\sum_{n=1}^{\infty}(-1)^{n}a_n$$

<u>**Definition**</u> : A series of the form

$\sum_{n=1}^{\infty}(-1)^{n}b_n$ or $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$, where $b_n > 0$ for all n, is called an alternating series, because the terms alternate between positive and negative values.

Example :

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \ldots$$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{2n+1} = \frac{1}{3} - \frac{2}{5} + \frac{3}{7} - \frac{4}{9} + \ldots$$

8.2 Convergence of Alternating Series

A famous result concerning alternating series is the **Alternating Series Test** (also known as the **Leibniz Test**). The test provides a criterion for determining the convergence of an alternating series.

Alternating Series Test (Leibniz Test)

An alternating series

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n$$

Converges if and only if the subsequent two requirements are met:

1. The sequence $\{a_n\}$ is monotonically decreasing, i.e., $a_{n+1} \leq a_n$ for all n sufficiently large.
2. The limit of the sequence $\{a_n\}$ is zero, i.e.,

$$\lim_{n\to\infty} a_n = 0$$

If both conditions are met, then the series converges.

Example 1

Consider the alternating harmonic series:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$$

1.The sequence $\left\{\frac{1}{n}\right\}$ is monotonically decreasing since $\frac{1}{n+1} \leq \frac{1}{n}$ for all n.

2. The limit of the sequence $\left\{\frac{1}{n}\right\}$ is zero, as $\lim_{n\to\infty}\frac{1}{n} = 0.$
Therefore, by the Alternating Series Test, the alternating harmonic series converges.

Conditional Convergence

A series that converges but does not converge absolutely is said to converge conditionally. The alternating harmonic series mentioned earlier is an example of a series that converges conditionally.

The Alternating Series Test: The series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n$$

converges if all three of the following three conditions are satisfied:

(1) $b_n > 0$ *for all* n;

(2) $b_{n+1} \leq b_n$ *for all* n;

(3) $\lim_{n\to\infty} b_n = 0$.

Example: The alternating series

$$\sum_{n=1}^{\infty}(-1)^n \frac{n}{n+1} = -\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \cdots$$

diverges. In this case

$b_n = \frac{n}{n+1}$.

We have

$$\lim_{n\to\infty} b_n = 1 \neq 0.$$

By the test for divergence, the series diverges.

8.3 Theorems on alternating Series

Theorem 1:

Statement: Let s be the sum of the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1} b_n$$

and let s_n be its nth partial sum. Suppose that

$$0 < b_{n+1} \leq b_n$$

for all n and

$$\lim_{n\to\infty} b_n = 0.$$

Then

$$|s - s_n| \leq b_{n+1}.$$

<u>Proof.</u> We have

$$s - s_n = (-1)^n b_{n+1} + (-1)^{n+1} b_{n+2}$$

$$+ (-1)^{n+2} b_{n+3} + (-1)^{n+3} b_{n+4} + \cdots$$

$$= (-1)^n (b_{n+1} - b_{n+2} + b_{n+3} - b_{n+4} + \cdots).$$

Since $0 < b_{n+1} \leq b_n$ for all n, we deduce that

$$|s - s_n| = b_{n+1} - b_{n+2} + b_{n+3} - b_{n+4} + \cdots \leq b_{n+1}.$$

Theorem 2:
Statement: (Leibniz Test) let

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

be an alternating series such that

i) $a_n > 0 \ \forall n = 1, 2, 3, \ldots$

ii) $a_1 \geq a_2 \geq a_3 \geq \ldots$, i.e., (a_n)

iii) $\lim_{n\to\infty} a_n = 0$

Proof: Let us first look at the even partial sums

$s_2, s_4, s_6, \ldots$ we have

$$s_2 = a_1 - a_2 \geq 0,$$
$$s_4 = s_2 + (a_3 - a_4) \geq s_2 \text{ because } a_3 \geq a_4,$$
$$s_6 = s_4 + (a_5 - a_6) \geq s_4 \text{ because } a_5 \geq a_6,$$

and, in general,

$$s_{2n+2} = s_{2n} + (a_{2n+1} - a_{2n+2}) \geq s_{2n} \text{ because } a_{2n+1} \geq a_{2n+2}$$

$$s_2 \leq s_4 \leq s_6 \leq \ldots\ldots$$

$$s_{2n} = a_1 - (a_2 - a_3) \ldots (a_{2n-2} - a_{2n-1}) - a_{2n}.$$

if $\varepsilon > 0 \ \exists K$ such that $n \geq K \Rightarrow |s_{2n} - s| \leq \frac{1}{2}\varepsilon$, and $a_{2n+1} \leq \frac{1}{2}\varepsilon$.

$$\sum (-1)^{n+1} a_n = s.$$

Now we check the inequality

$|s - s_n| < a_{n+1}.$

we note that

$$\left|(-1)^{n+2} a_{n+1} + \ldots + (-1)^{n+p+1} a_{n+p}\right| = \left|a_{n+1} - a_{n+2} + \ldots + a^{p-1} a_{n+9}\right|$$

The sum between absolute value signs can be expressed in the form

$$(a_{n+1} - a_{n+2}) + (a_{n+3} - a_{n+4}) + \ldots + \begin{cases} (a_{n+p-1} - a_{n+p}) & \text{of } p \text{ is even} \\ a_{n+p} & \text{if } p \text{ is odd} \end{cases}$$

The absolute value indication above can be deleted since the decreasing value of a indicates that the total is greater than zero. One way to express the total is as

$$a_{n+1} - (a_{n+2} - a_{n-3}) + \ldots \begin{cases} a_{n+p} & \text{if } p \text{ is even} \\ (a_{n+p-1} - a_{n+p)} & \text{if } p \text{ is odd} \end{cases}$$

This shows that the sum is

$\leq a_{n+1}.$

Hence the inequality

$|\mathrm{s} - \mathrm{s}_{\mathrm{n}}| < \mathrm{a}_{\mathrm{n+1}}$

holds. Hence the theorem.

Example 2:

Show that the series is convergent.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n+1}{n(n+1)} = \frac{3}{2} - \frac{5}{2.3} + \frac{7}{3.4} - \frac{9}{4.5} + \ldots$$

Solution:

For the given series we have,

$$a_n = \frac{2n+1}{n(n+1)}.$$

Clearly, $a_n > 0$ and

$$\lim_{n \to \infty} a_n = 0.$$

$$\frac{a_{n+1}}{a_n} = \frac{2n+3}{(n+1)(n+2)} \cdot \frac{n(n+1)}{2n+1} = \frac{2n^2 + 3n}{2n^2 + 5n + 2} < 1$$

Hence

$a_n > a_{n+1}$,

and therefore the series converges.

Example 2:

Use Ratio test to determine whether the following series are convergent.

i) $\sum_{n=1}^{\infty} \frac{n^3}{n!}$ ii) $\sum_{n=1}^{\infty} \frac{n^2 2^n}{n!}$

Solution :

(i)

Let $a_n = \frac{n^3}{n!}$, for $n = 1,2,\ldots$, so that

$$\frac{a_{n+1}}{a_n} = \left(\frac{(n+1)^3}{(n+1)!}\right) \times \left(\frac{n!}{n^3}\right)$$

$$= \frac{(n+1)^2}{n^3}$$

$$= \frac{n^3 + 2n + 1}{n^3} = \frac{1}{n} + \frac{2}{n^2} + \frac{1}{n^3}.$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0.$$

Hence it follows, from the Ratio Test that (i) is convergent.

ii) Let $a_n = \frac{n^2 2^n}{n!}$, for $n = 1,2,\ldots$, so that

$$\frac{a_{n+1}}{a_n} = \left(\frac{(n+1)^2 2^{n+1}}{(n+1)!}\right) \times \left(\frac{n!}{n^2 2^n}\right)$$

$$= \frac{2(n+1)}{n^2}$$

$$= 2\left(\frac{1}{n} + \frac{1}{n^2}\right).$$

Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$, it follows, from the Ratio Test that $\sum_{n=1}^{\infty} \frac{n^2 2^n}{n!}$ is convergent.

Example 3:

Test the convergence of the series whose general term is

$$\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}.$$

Solution:

$$\text{Let } u_n = \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{n^{3/2}}}$$

$$\text{Then } \lim_{n\to\infty}(u_n)^{1/n} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}}$$

$$= \frac{1}{\lim_{n\to\infty}\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}}$$

$$= \frac{1}{e} < 1.$$

Therefore series converges.

Example 4: Examine whether the given series diverges, converges conditionally, or converges absolutely or not

$$\sum_n \frac{(-1)n}{2^n},$$

Solution:

$$\text{Let } a_n = \frac{(-1)^n n}{2^n}. \text{Then } |a_n| = \frac{n}{2^n}$$

$$\frac{|a_{n+1}|}{|a_n|} = \left|\frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}}\right| = \frac{1}{2}\frac{n+1}{n}$$

$$\therefore \lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} \to \frac{1}{2} < 1$$

Because of its absolute convergence, the series is convergent. As an alternative, the lebeniz rule may be used to demonstrate the series' convergence.

Example 5: Examine the convergence and divergence of the following series

$$\sum_{n=1}^{\infty}\frac{n^2-3n+4}{5n^4-n}.$$

Solution: let

$$a_n=\frac{n^2-3n+4}{5n^4-n} \text{ and } b_n=\frac{1}{n^2}.$$

Then both a_n and b_n are positive

$$\frac{a_n}{b_n}=\frac{n^2-3n+4}{5n^4-n}\times\frac{n^2}{1}$$

$$=\frac{n^4-3n^3+4n^2}{5n^4-n}$$

$$=\frac{1-3n^{-1}+4n^{-2}}{5-n^{-3}}$$

Then

$$\lim_{n\to\infty}\frac{a_n}{a_n}\to\frac{1}{5} \text{ as } n\to\infty. \text{ Since } \frac{1}{5}\neq 0$$

and the series

$$\sum\frac{1}{n^2}$$

is convergent, the given series is convergent.

Example 5: Using the comparison test or the limit comparison test, Find the convergence of the following series

i) $\sum_{n=1}^{\infty}\frac{1}{n+\sqrt{n}}$ ii) $\sum_{n=1}^{\infty}\frac{\cos^2(2n)}{n^3}$

Solution:

$$a_n=\frac{1}{n+\sqrt{n}} \text{ and } b_n=\frac{1}{n}, \text{for } n=1,2,\ldots, \text{so that}$$

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{n}{n+\sqrt{n}}$$

$$=\lim_{n\to\infty}\frac{1}{1+\frac{1}{\sqrt{n}}}$$

$$=1\neq 0.$$

Since

$$\sum_{n=1}^{\infty}\frac{1}{n}\text{i}$$

n is divergent, we deduce, from the Limit Comparison Test, that

$$\sum_{n=1}^{\infty}\frac{1}{n+\sqrt{n}}$$

ii) We use the Comparison Test. Since

$$0 \le \cos^2(2n) \le 1, \text{ for } n = 1,2,...,$$

$$0 \le \frac{\cos^2(2n)}{n^3} \le \frac{1}{n^3}, \text{ for } n = 1,2,....$$

Since

$$\sum_{n=1}^{\infty}\frac{1}{n^3}$$

is convergent, we deduce, from the Comparison Test, that

$$\sum_{n=1}^{\infty}\frac{\cos^2(2n)}{n^3}$$

is convergent.

8.3 Summary

An alternating series is a series where terms alternate in sign, typically of the form $\sum_{n=1}^{\infty}(-1)^{n+1}a_n or \sum_{n=1}^{\infty}(-1)^n a_n$. Understanding focuses on the alternating series test (Leibniz's criterion) for convergence, which requires that terms decrease in absolute value and approach zero. Alternating series may converge conditionally or absolutely, with conditional convergence dependent on rearrangements. Applications include error estimation in numerical methods, series expansions in mathematics, and models in physics and finance. Mastery involves applying convergence criteria, manipulating series, and analyzing applications across disciplines.

8.4 Keywords

- Alternating Series
- Leibniz's Criterion
- Convergence

- Conditional Convergence
- Absolute Convergence
- Series
- Summation
- Rearrangement

8.5 Self Assessment

1. Consider the series$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$. Apply Leibniz's criterion to determine whether the series converges or diverges. Explain your reasoning.
2. Given the alternating series $\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n+1}$, find the sum of the series if it converges. Discuss any conditions under which the series converges.
3. In physics, a damped harmonic oscillator is described by the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin(\frac{n\pi x}{T})$, where T is the period of oscillation. Analyze the convergence of this series and its implications for understanding the oscillatory behavior of the system.
4. A mathematician investigates the alternating series $\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{2^n}$. Determine whether the series converges absolutely, conditionally, or diverges. Explain any implications for using this series in mathematical modeling.
5. An engineer analyzes an alternating series representation of a signal in a digital communication system:$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cos(2\pi xft)$, where f is the frequency of the signal. Discuss the conditions under which the series converges and how this analysis aids in signal processing applications.

8.6 Case Study

A mathematician explores the convergence of the alternating harmonic series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$.

Questions:

1. Determine whether the series converges conditionally or absolutely and justify your conclusion.

2. Discuss the historical significance of the alternating harmonic series in the development of mathematical analysis.
3. How might the mathematician extend the analysis to other alternating series and their properties?

8.7 References

- Royden, H., Fitzpatrick, P. (2018). Real Analysis. United Kingdom: Pearson.
- Trench, W. F. (2013). Introduction to Real Analysis. United Kingdom: Prentice Hall/Pearson Education

Unit – 9

Test for Convergence

Learning Objective

- Understand the ratio test and its application to determine convergence by examining the limit of the ratio of consecutive terms.
- Analyze how rearranging terms in a series affects its convergence and how rearrangements can preserve or change convergence properties.
- Develop the ability to analyze series convergence using multiple tests and determine the most appropriate test for a given series.

Structure

9.1 Necessary Conditions For Convergent Series

For every convergent series Σu_n,

$$\lim_{n\to\infty} u_n = 0$$

Solution:

$$\text{Let} \quad S_n = u_1 + u_2 + u_3 + \ldots + u_n$$

$$\lim_{n\to\infty} S_n = k$$

$$\lim_{n\to\infty} S_{n-1} = k$$

$$S_n = S_{n-1} + u_n$$

$$u_n = S_n - S_{n-1}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} [S_n - S_{n-1}] = 0$$

$$\lim_{n\to\infty} u_n = 0$$

Corollary: Converse of the above theorem is not true.

e.g., $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots + \frac{1}{\sqrt{n}} + \ldots \infty$ is divergent.

$$S_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots + \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \ldots + \frac{1}{\sqrt{n}}$$

$$> \frac{n}{\sqrt{n}} > \sqrt{n}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sqrt{n} = \infty$$

9.2 Comparison Test

Consider two positive terms Σu_n, and Σv_n, be such that

$$\lim_{n\to\infty} \frac{u_n}{v_n} = k$$

(finite number), then both series converge or diverge together.

Proof: There exists a positive number ε, however small, such that

$$\left|\frac{u_n}{v_n}-k\right|<\varepsilon \text{ for } n>m \qquad i.e., \; -\varepsilon<\frac{u_n}{v_n}-k<+\varepsilon$$

$$k-\varepsilon<\frac{u_n}{v_n}<k+\varepsilon \text{ for } n>m$$

$$k-\varepsilon<\frac{u_n}{v_n}<k+\varepsilon \text{ for all } n. \qquad ...(1)$$

Case 1: Σu_n is divergent then

$$\lim_{n\to\infty}(v_1+v_2+\ldots+v_n) = h \text{ (say)}$$

From (1), $u_n<(k+\varepsilon)\,v_n$ for all n.

$$\lim_{n\to\infty}(u_1+u_2+\ldots+u_n) < (k+\varepsilon)\lim_{n\to\infty}(v_1+v_2+\ldots+v_n)=(k+\varepsilon)\,h$$

Hence, Σu_n is also Convergent

Case 2: Σv_n is divergent then

$$\lim_{n\to\infty}(v_1+v_2+\ldots+v_n) \to \infty \qquad ...(2)$$

Now from (1) $\qquad k-\varepsilon<\frac{u_n}{v_n}$

$$u_n>(k-\varepsilon)\,v_n \text{ for all } n$$

$$\lim_{n\to\infty}(u_1+u_2+\ldots+u_n) > (k-\varepsilon)\lim_{n\to\infty}(v_1+v_2+\ldots+v_n)$$

From (2) $\qquad \lim_{n\to\infty}(u_1+u_2+\ldots+u_n)\to\infty$

Hence, Σu_n is also divergent

Example 1: Test the series for Convergence or divergence

$$\sum_{n=1}^{\infty}\frac{1}{n+10}$$

Solution:

$$u_n=\frac{1}{n+10}$$

$$v_n=\frac{1}{n}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{n}{n+10}=\lim_{n\to\infty}\frac{1}{1+\frac{10}{n}}=1=\text{finite number.}$$

Hence, Σu_n is also divergent.

Example 2: Test whether the given series is convergent or not.

$$\frac{1}{1\cdot 2\cdot 3}+\frac{3}{2\cdot 3\cdot 4}+\frac{5}{3\cdot 4\cdot 5}+\dots\infty$$

Solution

$$u_n=\frac{2n-1}{n\,(n+1)\,(n+2)}=\frac{1}{n^2}\,\frac{2-\frac{1}{n}}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{2-\frac{1}{n}}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}=2=\text{ finite number.}$$

Since Σu_n, is also convergent.

9.3 D'alembert's Ratio Test

Statement: ConsiderΣu_n, is a positive term series such that

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=k$$

(i) the series is convergent if $k<1$

(ii) the series is divergent if $k>1$

Example 3: Test for convergence the series whose nth term is $n^2/2^n$

Solution

$$u_n=\frac{n^2}{2^n},\quad u_{n+1}=\frac{(n+1)^2}{2^{n+1}}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{(n+1)^2}{2^{n+1}}\cdot\frac{2^n}{n^2}=\lim_{n\to\infty}\frac{1}{2}\left(1+\frac{1}{n}\right)^2=\frac{1}{2}<1$$

Hence the Series is Convergent

Example 4: Test for convergence the series whose nth term is $2^n/n^3$

Solution

$$u_n = \frac{2^n}{n^3}, \; u_{n+1} = \frac{2^{n+1}}{(n+1)^3}$$

$$\frac{u_{n+1}}{u_n} = \frac{2^{n+1}}{(n+1)^3} \cdot \frac{n^3}{2^n} = \frac{2}{\left(1+\frac{1}{n}\right)^3} \qquad \lim_{n \to \infty} \frac{u_{n+1}}{u_n} = \lim_{n \to \infty} \frac{2}{\left(1+\frac{1}{n}\right)^3} = 2 > 1$$

Hence the series is divergent

9.4 Raabe's Test (Higher Ratio Test)

If Σu_n, is a positive term series such that

$$\lim_{n \to \infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = k,$$

the series is convergent if k>1 and the series is divergent if k < 1.

Proof:

Case I. k >1

Let p be such that k>p> 1 and compare the given series Σu_n, with $\Sigma 1/n^p$ which is convergent as p > 1.

$$\frac{u_n}{u_{n+1}} > \frac{(n+1)^p}{n^p} \quad \text{or} \quad \left(\frac{u_n}{u_{n+1}}\right) > \left(1+\frac{1}{n}\right)^p > 1 + \frac{p}{n} + \frac{p(p-1)}{\lfloor 2} \frac{1}{n^2} + \ldots$$

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) > p + \frac{p(p-1)}{n!} \frac{1}{n^2} + \ldots$$

$$\lim_{n \to \infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) > p$$

and k>p which is true as k>p>1; Σu_n, is convergent when k> 1

Case II. k< 1

Same steps as in Case 1.

Example 4: Test whether the given series is convergent or not.

$$\frac{x}{1.2} + \frac{x^2}{3.4} + \frac{x^3}{5.6} + \frac{x^4}{7.8} + \ldots.$$

Solution

$$u_n = \frac{x^n}{(2n-1)\,2n} \text{ and } u_{n+1} = \frac{x^{n+1}}{(2n+1)\,(2n+2)}$$

$$\frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{(2n+1)\,(2n+2)} \times \frac{(2n-1)2n}{x^n} = \frac{x\left(1-\frac{1}{2n}\right)}{\left(1+\frac{1}{2n}\right)\left(1+\frac{2}{2n}\right)},$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = x$$

(i) If x<1, Σu_n is convergent (ii) If x>1, Σu_n, is divergent (iii) If x=1, Test fails Let us apply Rabee's Test when x=1

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left[\frac{(2n+1)\,(2n+2)}{2n\,(2n-1)} - 1\right] = \lim_{n\to\infty} n\left[\frac{(2n+1)\,(2n+2) - 2n\,(2n-1)}{2n\,(2n-1)}\right]$$

$$= \lim_{n\to\infty} n\left[\frac{(8n+2)}{(2n)\,(2n-1)}\right] = \lim_{n\to\infty} \frac{\left(2+\frac{2}{4n}\right)}{1\left(1-\frac{1}{2n}\right)} = 2$$

So the Series is Convergent

9.5 Summary

1. The test for convergence of positive term series assesses whether an infinite series $\sum_{n=1}^{\infty} a_n$ where $a_n \geq 0$, converges or diverges. Methods include:
2. **Comparison Test**: Compares with a known convergent or divergent series.
3. **Limit Comparison Test**: Compares with a simpler series using limits.
4. **Integral Test**: Relates convergence to an associated improper integral.
5. **Ratio Test**: Analyzes convergence by evaluating the limit of the ratio of consecutive terms.

9.6 Keywords

- Series Convergence
- Comparison Test

- Limit Comparison Test
- Integral Test
- Ratio Test
- Convergence Criteria
- Convergence Tests

9.7 Self Assessment

1. Use the comparison test to determine the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$. Identify a suitable series for comparison and justify your conclusion.
2. Apply the limit comparison test to analyze the convergence of $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$. Choose an appropriate series for comparison and explain the result.
3. Evaluate the convergence of $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ using the integral test. Set up the corresponding improper integral, compute its value, and interpret the result.
4. Use the ratio test to determine whether $\sum_{n=1}^{\infty} \frac{n}{2^n}$converges or diverges. Show the steps involved in applying the ratio test and interpret the limit.
5. An engineer models the decay of a vibrating system with the series $\sum_{n=1}^{\infty} \frac{e^{-n}}{n}$. Discuss the convergence of this series using a suitable convergence test and explain its relevance in engineering applications.

9.8 Case Study

A mathematician investigates the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

Questions:

1. Use the ratio test to determine the convergence of the series. Show the steps involved in applying the ratio test and interpret the result.
2. Discuss the relevance of the convergence result in mathematical modeling and its applications in real-world scenarios.
3. How might the mathematician extend this analysis to other similar series and their properties?

9.9 References

- Royden, H., Fitzpatrick, P. (2018). Real Analysis. United Kingdom: Pearson.
- Trench, W. F. (2013). Introduction to Real Analysis. United Kingdom: Prentice Hall/Pearson Education

Unit – 10

Cauchy’s Test

Learning Objective:

- Identify scenarios where the Cauchy nth root test is particularly useful and where it may not apply effectively.
- Apply the Cauchy nth root test in mathematical models, such as power series representations and recursive sequences.
- Understand how the test is used in scientific fields, such as physics (Fourier series) and engineering (signal processing), to model and analyze real-world phenomena.

Structure

10.1 Cauchy's Root Test

Statement : Consider Σu_n, is a positive term series such that

$$\lim_{n \to \infty} (u_n)^{1/n} = k,$$

(i) if $k<1$, series converges.

(ii) if $k> 1$, series diverges.

$$\left|(u_n)^{1/n} - k\right| < \varepsilon \text{ for } n > m$$
$$k - \varepsilon < (u_n)^{1/n} < k + \varepsilon \text{ for } n > m$$

(i)
$$k < 1$$
$$k + \varepsilon < r < 1$$
$$(u_n)^{1/n} < k$$

$\Rightarrow$
$$u_n < k^n$$
$$u_1 + u_2 + \ldots \infty < k + k^2 + \ldots + k^n + \ldots \infty$$
$$< \frac{1}{1-k}$$

The Series is Convergent

(ii)
$$k > 1$$
$$k - \varepsilon > 1$$
$$(u_n)^{1/n} > k - \varepsilon > 1$$
$$u_n > 1$$
$$S_n = u_1 + u_2 + \ldots + u_n > n$$
$$\lim_{n \to \infty} S_n \to \infty$$

So the Series is Divergent

(iii) $k = 1$

If $\lim_{n \to \infty} (u_n)^{1/n} = 1$, the test fails.

For example:

$$\Sigma u_n = \sum \frac{1}{n^p}$$

$$\lim_{n \to \infty} (u_n)^{1/n} = \lim_{n \to \infty} \left(\frac{1}{n^p}\right)^{1/n} = \lim_{n \to \infty} \left(\frac{1}{n^{1/n}}\right)^{-p} = 1 \text{ for all } p,\ k = 1$$

But $\Sigma\ 1/n^p$ is convergent for p>1 and divergent for p≤1.

Example1: Find the convergence of the series

$$\sum \frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$$

Solution

$$u_n = \frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}, \quad (u_n)^{1/n} = \left[\frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}\right]^{\frac{1}{n}} = \frac{1}{\left(1+\frac{1}{n}\right)^{n}}$$

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{n}} = \frac{1}{e} < 1$$

Example 2: Discuss the Convergence of the following series

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3} + \ldots \infty$$

Solution

$$u_n = \left[\frac{(n+1)^{n+1}}{n^{n+1}} - \frac{(n+1)}{n}\right]^{-n}$$

$$[u_n]^{1/n} = \left[\left[\frac{(n+1)^{n+1}}{n^{n+1}} - \frac{n+1}{n}\right]^{-n}\right]^{\frac{1}{n}} = \left[\frac{(n+1)^{n+1}}{n^{n+1}} - \frac{n+1}{n}\right]^{-1}$$

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \left[\left(1+\frac{1}{n}\right)^{n+1} - \left(1+\frac{1}{n}\right)\right]^{-1} = (e-1)^{-1} = \frac{1}{e-1} < 1$$

Hence the given series is convergent

10.2 Cauchy's Integral Test

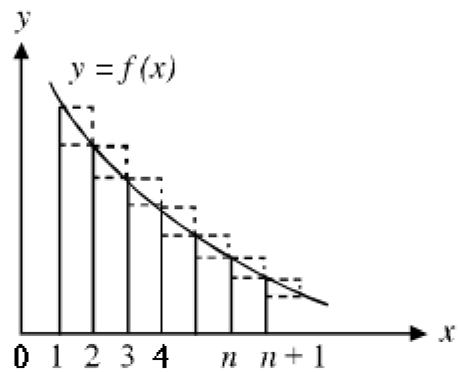
Statement: A positive term series f(1)+f(2)+f(3) + ... +f (n) + ...

where, depending on the integral, f (n) either converges or diverges as n rises.

$$\int_1^\infty f(x)\,dx$$

is finite or infinite.

Proof:



In the figure, the area under the curve in the image spans the interval between the sum of the areas of small and big rectangles, from x = 1 to x = n + 1.

$$\Rightarrow f(1)+f(2)+\ldots+f(n) \geq \int_{1}^{n+1} f(x)\,dx \geq f(2)+f(3)+\ldots+f(n+1)$$

$$S_n \geq \int_{1}^{n+1} f(x)\,dx \geq S_{n+1}-f(1)$$

Since the second inequality states that lim S+1 is also finite if the integral has a finite value, Σf(n) is convergent as n approaches ∞.

Likewise, in the event that the integral is infinite, the series is divergent as the first inequality states that limSn → ∞.

Example 3: Use the integral test to find the convergence of the p-series

$$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots+\frac{1}{n^p}+\ldots\infty$$

Solution

(i) When p >1

$$\int_1^\infty f(x)\,dx = \lim_{m\to\infty}\int_1^m \frac{1}{x^p}dx = \lim_{m\to\infty}\left[\frac{x^{1-p}}{1-p}\right]_1^m = \lim_{m\to\infty}\frac{1}{1-p}[m^{1-p}-1]$$

$$= \lim_{m\to\infty}\frac{1}{1-p}\left[\frac{1}{m^{p-1}}-1\right] = \frac{1}{p-1}, \text{ which is finite.}$$

The Series is Convergent

(ii) When p <1

$$\int_1^\infty f(x)\,dx = \frac{1}{1-p}\left[\lim_{m\to\infty}(m^{1-p}-1)\right] \to \infty$$

The Series is divergent

(iii) When p = 1

$$\int_1^\infty \frac{1}{x}dx = [\log x]_1^\infty \to \infty$$

The Series is Convergent

Example 4: Examine the convergence of the series

$$\sum_{n=2}^{\infty} \frac{1}{n \log n}$$

Solution

$$f(x) = \frac{1}{x\log x}$$

$$\int_2^\infty \frac{1}{x\log x}dx = \lim_{m\to\infty}[\log\log x]_2^m = \lim_{m\to\infty}[\log\log m - \log\log 2] \to \infty$$

10.3 Cauchy's Condensation Test

If a_n is a positive integer greater than 1 and ϕ(n) is positive for all positive integral values of n, then the two series, Σ ∅ (n) and Σa_n∅ (an), are either both divergent or both convergent.

Example 5: Examine the convergence

$$1+\frac{1}{2\,(\log 2)^{p}}+\frac{1}{3\,(\log 3)^{p}}+\ldots+\frac{1}{n\,(\log n)^{p}}+\ldots$$

Solution

$$\phi(n) = \frac{1}{n\,(\log n)^{p}}$$

$$a^{n}\left[\frac{1}{a^{n}\,(\log a^{n})^{p}}\right] \quad i.e., \quad \frac{1}{(\log a^{n})^{p}} \; i.e., \; \frac{1}{(n\log a)^{p}} \; i.e., \; \frac{1}{(\log a)^{p}}\times\frac{1}{n^{p}}$$

the given series will be convergent or divergent if $\sum\left[\frac{1}{(\log a)^{p}}\times\frac{1}{n^{p}}\right]$ is convergent or divergent, *i.e.*, if $\sum\frac{1}{n^{p}}$ is convergent or divergent.

But we know that $\Sigma 1/n^{p}$ is convergent when p> 1 and divergent when p = 1 or < 1.

10.4 Summary

The Cauchy nth root test is a convergence criterion for infinite series that states if $\lim_{n\to\infty}\sqrt[n]{a_n} = L$, then:

- If L<1, the series $\sum_{n=1}^{\infty} a_n$an converges absolutely.
- If L>1 or L=∞, the series diverges.
- If L=1, the test is inconclusive. The test assesses convergence by evaluating the limit of the nth root of the absolute values of the terms a_n. It is particularly useful for determining absolute convergence of series with positive terms, providing a straightforward criterion for convergence analysis.

10.5 Keywords

- Cauchy's Condensation Test
- Series
- Convergence
- Divergence
- Cauchy's Root Test
- Cauchy's Integral Test

10.6Self Assessment

1. Apply the Cauchy nth root test to determine the convergence of the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$. Calculate $\lim_{n\to\infty} \sqrt[n]{\frac{n^n}{n!}}$and interpret the result.
2. Compare the Cauchy nth root test with the ratio test for the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$. Determine which test is more suitable for evaluating the convergence of this series and justify your answer.
3. An engineer analyzes the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ in the context of signal processing. Use the Cauchy nth root test to determine the convergence of the series and discuss its implications for engineering applications.
4. A physicist studies wave behavior using the series $\sum_{n=1}^{\infty} \frac{\sin(n\pi x)}{\mathrm{n}^2}$, where x represents position along a vibrating string. Apply the Cauchy nth root test to analyze the convergence of this series and interpret the result in the context of wave mechanics.
5. Investigate the convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{\mathrm{n}^\mathrm{n}}$ using the Cauchy nth root test. Calculate $\lim_{n\to\infty} \sqrt[n]{\frac{(-1)^n n!}{\mathrm{n}^\mathrm{n}}}$ and discuss the convergence criteria based on the test's findings.

10.7 Case Study

A mathematician investigates the convergence of the series $\sum_{n=1}^{\infty} \frac{n!}{\mathrm{n}^\mathrm{n}}$ using the Cauchy nth root test.

Questions:

1. Apply the test to determine the convergence of the series and calculate $\lim_{n\to\infty} \sqrt[n]{\left|\frac{n!}{\mathrm{n}^\mathrm{n}}\right|}$
2. Discuss the theoretical implications of the test's findings in mathematical analysis and its relevance to series with factorial growth.
3. How might the mathematician extend this analysis to other series involving factorial terms or exponential growth?

10.8 References

- Royden, H., Fitzpatrick, P. (2018). Real Analysis. United Kingdom: Pearson.
- Trench, W. F. (2013). Introduction to Real Analysis. United Kingdom: Prentice Hall/Pearson Education.